

Why FARIMA Models are Brittle

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Abstract

The FARIMA models, which have long-range-dependence (LRD), are widely used in many areas. Through deriving a precise characterisation of the spectrum, autocovariance function, and variance time function, we show that this family is very atypical among LRD processes, being extremely close to the fractional Gaussian noise in a precise sense. Furthermore, we show that this closeness property is not robust to additive noise. We argue that the use of FARIMA, and more generally fractionally differenced time series, should be reassessed in some contexts, in particular when convergence rate under rescaling is important and noise is expected.

Keywords: FARIMA, fractionally differenced process, self-similarity, fGn, long-range dependence, Hurst parameter

1 Introduction

For a wide variety of purposes including data modelling, synthetic data generation, and the testing of statistical estimators, tractable and flexible time series models are indispensable. The well known *AutoRegressive Moving Average* (ARMA) family, for example, allows for a wide variety of short range correlation structures, and has been used in many contexts.

Long-Range Dependence (LRD), or long memory, in stationary time series is a phenomenon of great importance Taqqu (2002). The *Fractional AutoRegressive Integrated Moving Average* (FARIMA) models Hosking (1981); Granger and Joyeux (1980) are very widely used as a class which inherits the advantages of ARMA, while exhibiting LRD with tunable *Hurst parameter*, the scaling parameter of LRD. They have in particular been widely used to parsimoniously model data sets exhibiting LRD (for example Ilow (2000)), and more importantly for our purposes here, they have also been employed to make quantitative assessments of the behaviour of stochastic systems in the face of LRD (for example Barbe and McCormick (2010)).

A good example is in relation to estimators of the Hurst parameter H . FARIMA models have been used (for example Taqqu et al. (1995); Taqqu and Teverovsky (1997); Abry et al. (2003)) in order to evaluate the performance of H estimators under circumstances more challenging than that of the canonical *fractional Gaussian Noise* (fGn), in particular to assess small sample size performance using Monte Carlo simulation. Although explicit claims of the generality of the FARIMA family are not made, implicitly it is taken to be a typical class of LRD time series in some sense, and so results obtained using it are taken to be representative for LRD inputs in general.

In fact, no parametric model can be truly typical. However, for a model class to be useful it should be representative for the purposes to which it is commonly put. In this paper, we show that FARIMA time series,

and more generally time series whose LRD scaling derives directly from fractional differencing such as the FEXP models Robinson (1994), are far from typical when it comes to their LRD character, the very quality for which they were first introduced. In a sense we make precise, out of all possible LRD time series, their LRD behaviour is in fact ‘as close as possible’ to that of fGn. A key technical consequence is ultra-rapid convergence to fGn under the rescaling operation of aggregation. The implications for the role of the family is strong, namely that, in regards to LRD behaviour, *FARIMA offers no meaningful diversity beyond fGn*. A second key consequence is that the addition of additive noise (of almost any kind) pushes a FARIMA process out of the immediate neighbourhood of fGn, changing the convergence rate. In other words FARIMA is structurally unstable in this sense or *brittle*, and is therefore unsuited for use as a class of LRD time series representing real-world signals.

This work arose out of our prior study of (second-order) self-similarity of stationary time-series Gefferth et al. (2003), which highlighted the benefits of the variance time function (VTF) formulation of the autocovariance structure, over the more commonly used autocovariance function (ACVF) formulation. Using the VTF, questions of process convergence under rescaling to exactly (second-order) self-similar limits can often be more simply stated and studied.

The paper is structured as follows. After Section 2 on background material, Section 3 establishes the main results. It begins by characterising a link between a fractionally differenced process and fGn in the spectral domain. Using it, we prove that related Fourier coefficients in the time domain decay extremely quickly, and then show that as a result the VTFs of the fractionally differenced process and fGn are extremely close. We then explain why this behaviour is so atypical, and how it results in fast convergence to fGn. Finally we go on to provide distinct direct proofs of closely related results for the ACVF and spectral formulation which are of independent interest. In particular, they lead to additional closeness results for the spectrum. In Section 5 we explain why fractional processes are not robust to the addition of additive noise, even noise of particularly non-intrusive character. We also provide numerical illustrations of this brittleness, and of the fast convergence to fGn of FARIMA processes. We conclude and discuss possible implications of our findings in Section 6.

Very early versions of this work appear in the 2002 workshop papers Gefferth et al. (2002); Gefferth et al. Nov (2002).

2 Background

Let $\{X(t), t \in \mathbf{Z}\}$ denote a discrete time second-order stationary stochastic process. The mean μ and variance $\mathcal{V} > 0$ of such a process are independent of t , and the *autocovariance function* (ACVF), $\gamma(k) := E[(X(t) - \mu)(X(t+k) - \mu)]$, depends only on the lag k , $k \in \mathbf{Z}$, and $\gamma(k) = \gamma(-k)$.

A description of the autocovariance structure which is entirely equivalent to γ is the variance time function, defined as $\omega(n) = (\mathbf{I}\gamma)(n) := \sum_{k=0}^{n-1} \sum_{i=-k}^k \gamma(i)$ $n = 1, 2, 3, \dots$, where \mathbf{I} denotes the double integration operator acting on sequences. Its normalised form, the *correlation time function* (CTF), is just $\phi(n) = \omega(n)/\omega(1) = \omega(n)/\mathcal{V}$. In terms of the original process, $\omega(n)$ is just the variance of the sum $\sum_{t=1}^n X(t)$. It is convenient to symmetrically extend ω and ϕ to \mathbf{Z} by setting $\omega(n) := \omega(-n)$ for $n < 0$ and $\omega(0) = 0$.

2.1 LRD, Second-Order Self-Similarity, and Comparing to fGn

There are a number of definitions of long-range dependence, all of which encapsulate the idea of slow decay of correlations over time. Common definitions include power-law tail decay of the ACVF $\gamma(n) \stackrel{n \rightarrow \infty}{\sim} c_\gamma n^{2H-2}$, or power-law divergence of the spectral density at the origin $f(x) \stackrel{x \rightarrow 0}{\sim} c_f |x|^{-(2H-1)}$ for related constants c_γ and c_f (see for example Taqqu (2002), Section 4).

The well known *fractional Gaussian noise* (fGn) family, parameterised by the *Hurst parameter* $H \in [0, 1]$ and variance $\mathcal{V} > 0$, has $\omega(m) = \omega_{H,\mathcal{V}}^*(m) := \mathcal{V} m^{2H}$ (to lighten notation we sometimes write ω_H^* or simply ω^*). It has long memory if and only if $H \in (1/2, 1]$.

In this paper we compare against fGn with $H \in (1/2, 1]$ as it plays a special role among among LRD processes; that of being a family of *second-order self-similar* time series¹. To understand how this comparison can be made, we must define self-similarity and related notions.

Self-similarity relates to invariance with respect to a rescaling operation. In the present context, the time rescaling is provided by what is commonly called *aggregation*. For a fixed $m \geq 1$, the *aggregation of level m* of the original process X is the process $X^{(m)}$ defined as

$$X^{(m)}(t) := \frac{1}{m} \sum_{j=m(t-1)+1}^{mt} X(j).$$

The γ , ω , ϕ functions and the variance of the m -aggregated process will be denoted by $\gamma^{(m)}$, $\omega^{(m)}$, $\phi^{(m)}$ and $\mathcal{V}^{(m)}$ respectively. It is not difficult to show Gefferth et al. (2003) that

$$\omega^{(m)}(n) = \frac{\omega(mn)}{m^2}, \quad \mathcal{V}^{(m)} = \frac{\omega(m)}{m^2}. \quad (1)$$

To seek invariance, the time rescaling must be accompanied by a compensating amplitude rescaling. This is performed naturally by dividing by $\mathcal{V}^{(m)}$, which amounts to examining the effect of aggregation on the correlation structure. Combining the time and amplitude rescalings yields the correlation renormalisation

$$\phi^{(m)}(n) = \frac{\phi(mn)}{\phi(m)} = \frac{\omega(mn)}{\omega(m)}. \quad (2)$$

We can now define second-order self-similarity as the fixed points of this operator.

Definition 1. A process is second-order self-similar iff $\phi^{(m)} = \phi$, for all $m = 1, 2, 3, \dots$

It is easy to see that fGn, which has $\phi(m) = \phi_H^*(m) := m^{2H}$, satisfies this definition for all $H \in [0, 1]$.

Given a fixed point $\phi_H^*(n)$, we define its *domain of attraction* (DoA) to be those time series which converge to it pointwise under the action of (2). This definition is very general, in particular it includes processes whose VTF's have divergent slowly varying prefactors, as these cancel following normalization (see Section 3.3). It provides a natural way to define LRD which subsumes and generalises most other definitions including those above Gefferth et al. (2003): *a time series is long-range dependent if and only if it is in the domain of attraction of $\phi_H^*(n)$ for some $H \in (0.5, 1]$.*

With the above definitions the DoA are revealed as the natural way to partition the space of all LRD processes, namely into sets of processes each corresponding to the same unique normalized fGn fixed point. Since all processes within a DoA converge to the same fixed point, their asymptotic structure can be meaningfully compared both against each other and to the fixed point itself. Alternatively if two processes were in different DoA's then they cannot be close asymptotically as they would converge to different processes. Section 3.2 provides a precise characterisation of the closeness of a fractionally differenced process to its corresponding fixed point, and its associated fast convergence under renormalization.

Within a given DoA, one can further partition processes according to some measure of distance from the common fixed point. Section 3.3 establishes such a notion, enabling a comparison of this closeness to that of other members of the DoA to be made.

2.2 Fractionally Differenced Processes and FARIMA

Let B denote the backshift operator. The fractional differencing operator of order $d > -1$ is given by

$$(1 - B)^d := \sum_{j=0}^{\infty} \Gamma(j - d) / \Gamma(-d) \Gamma(j + 1) B^j.$$

¹Until recently, fGn was considered to be the only such family. A second (and final) family was discovered recently Gefferth et al. (2004).

Let $\{Y(t), t \in \mathbf{Z}\}$ be a second-order stationary stochastic process. Assuming $H \in (0, 1)$ the process

$$X := (1 - B)^{-(H-1/2)} Y$$

is called a fractionally differenced process with differencing parameter $H - 1/2$ driven by Y .

If h is the spectral density of Y then X has spectral density (Brockwell and Davis (1991), Thm. 4.10.1)

$$f_H(x) = h(x) |1 - e^{2\pi i x}|^{-(2H-1)} = h(x) |2 \sin \pi x|^{-(2H-1)}, \quad x \in [-1/2, 1/2]. \quad (3)$$

In this paper we assume that Y is short-range dependent, and in particular that h satisfies:

- $h(x) > 0$ and is continuous for all $x \in [-1/2, 1/2]$ (and is therefore bounded);
- h is three times continuously differentiable on $(-1/2, 1/2)$ (and is therefore in C^3).

Under such conditions, the ACVF of X exists and satisfies $\gamma_H(n) \sim c_\gamma n^{2H-2}$ for some constant c_γ (Brockwell and Davis (1991), Thm. 13.2.2). Hence, when $H \in (1/2, 1)$ the process X is LRD with Hurst parameter H .

An important example of a fractionally differenced process is the FARIMA class Hosking (1981) where h is the spectral density of a causal invertible ARMA model. This family includes the ARMA family as the special case $H = 1/2$. Another class is the class of FEXP-models (e.g. Bloomfield (1973); Robinson (1994); Beran (1993)) which comes from taking the logarithm of h to be a trigonometric polynomial, i.e. $\log h(x) = \theta_1 \cos x + \theta_2 \cos(2x) + \dots + \theta_{q-1} \cos((q-1)x)$ for real coefficients. Both FARIMA and FEXP models are widely used in statistical applications since, in addition to exhibiting LRD, they both enable modelling of arbitrary short-range correlation structures.

2.3 Normalizing a Fractionally Differenced Process to its fGn Limit

To identify the fGn fixed point of a fractionally differenced time series only the value of H need be determined. When aggregating an unnormalised fractionally differenced time series however, to identify the corresponding limiting fGn time series we must in addition know the correct variance \mathcal{V} . The purpose of this section is to define notation to make this simple and along the way to provide useful expressions for the spectra of these processes.

The ACVF, VTF, and spectral density corresponding to the fixed point are denoted γ_H^* , ω_H^* , and f_H^* , respectively. The latter is given by (see Samorodnitsky and Taqqu (1994))

$$\begin{aligned} f_H^*(x) &= c_f^* \pi^{-2} (2\pi)^{2H+1} \sin^2(\pi x) \sum_{j=-\infty}^{\infty} |2\pi j + 2\pi x|^{-(2H+1)} \\ &\stackrel{x \rightarrow 0}{\sim} c_f^* |x|^{-(2H-1)}, \quad x \in [-1/2, 1/2], \end{aligned} \quad (4)$$

where $c_f^* = \mathcal{V}(2\pi)^{2-2H} C(H) > 0$ is the prefactor of the power-law at the origin, and $C(H) = \pi^{-1} H \Gamma(2H) \sin(H\pi)$ (see Samorodnitsky and Taqqu (1994), pp 333-4, but note that the change to normalised frequency multiplies f_H^* by 2π , and c_f^* by $(2\pi)^{2-2H}$).

We denote by γ_H , ω_H and f_H the ACVF, VTF and spectral density of a fractional process with Hurst parameter $H \in (1/2, 1)$. In view of (3), the latter is given by

$$\begin{aligned} f_H(x) &= h(x) |2 \sin \pi x|^{-(2H-1)} = c_f (2\pi)^{2H-1} \frac{h(x)}{h(0)} |2 \sin \pi x|^{-(2H-1)} \\ &\stackrel{x \rightarrow 0}{\sim} c_f |x|^{-(2H-1)}, \quad x \in [-1/2, 1/2], \end{aligned} \quad (5)$$

where $c_f = (2\pi)^{1-2H} h(0) > 0$. In the case of a pure fractionally differenced process, such as FARIMA(0, d , 0), $h(x) = h(0) = 2\pi$, and $c_f = (2\pi)^{2-2H}$ (note again the changes related to normalised frequency, in particular the factor of 2π is built into $h(0)$).

To conclude, the particular fGn to which the fractionally differenced process will converge under renormalisation is the one such that $c_f^* = c_f$. From this, the value of \mathcal{V} can be obtained using the expressions for c_f^* and c_f above, if needed.

2.4 Regularity and Other Notations

Denote for $\alpha \geq 0$ by Λ_α the normed space of uniformly α -Hölder continuous functions on $[-1/2, 1/2]$,

$$\Lambda_\alpha := \{\varphi: [-1/2, 1/2] \rightarrow \mathbf{R} : \|\varphi\|_{\Lambda_\alpha} < \infty\},$$

where $\|\cdot\|_{\Lambda_\alpha}$ is the α -Hölder norm

$$\|\varphi\|_{\Lambda_\alpha} := \sup_{x, y \in [-1/2, 1/2]} |\varphi(x) - \varphi(y)| |x - y|^{-\alpha}.$$

Hence $\Lambda_\alpha \supseteq \Lambda_\beta$ whenever $\alpha \leq \beta$. The space Λ_α is closed under pointwise multiplication, addition, and composition with functions in Λ_1 . In particular, the subset of Λ_α whose members are bounded away from zero is closed under reciprocation (i.e. if $g \in \Lambda_\alpha$, and g is bounded away from zero, then so is $1/g$). Observe that $\varphi \in \Lambda_1$ whenever φ' exists and is bounded. Functions in Λ_α are absolutely continuous.

The linear space of functions of bounded variation on $[-1/2, 1/2]$, denoted V , is defined by

$$V := \{\varphi: [-1/2, 1/2] \rightarrow \mathbf{R} : \|\varphi\|_V < \infty\},$$

where $\|\cdot\|_V$ is the total variation norm

$$\|\varphi\|_V := \sup \left\{ \sum_{i=1}^{|P|} |\varphi(x_i) - \varphi(x_{i-1})| : P = \{x_0, x_1, \dots, x_n\} \text{ is a partition of } [-1/2, 1/2] \right\}.$$

V is also closed under pointwise multiplication and addition (Apostol (1974), Thm. 6.9), and reciprocation of those functions in V bounded away from zero (Apostol (1974), Thm. 6.10). Any differentiable function with bounded derivative on $(-1/2, 1/2)$ is of bounded variation on $[-1/2, 1/2]$ (Apostol (1974), Thm. 6.6).

We shall use the notation \star for convolution of sequences. For sequences a and b

$$(a \star b)_n = \sum_{j=-\infty}^{\infty} a_j b_{n-j}, \quad n \in \mathbf{Z}.$$

The convolution is said to exist if the infinite sum converges for all n . When needed for clarity, we also use $(a \star b)(n)$ to denote $(a \star b)_n$.

Throughout, by *smooth function* we mean one in C^∞ .

3 Fractionally Differenced Processes are Not Typical LRD Processes

The goal of this section is to establish our main results, rigorous characterisations of the closeness of the asymptotic covariance structure of a fractionally differenced process to that of fGn.

Our approach is simple and can be described as follows. We begin in the spectral domain where the relationship between the processes can be simply stated through a function g by defining

$$f_H(x) = f_H^*(x)g(x). \tag{6}$$

The simple closed form of the spectra (4) and (5) allow g to be explicitly written. We study the properties of g , obtaining a characterisation of the closeness of the processes in the spectral domain (Theorem 1). This leads to a convolution formulation $\gamma_H = \gamma_H^* \star G$ in the time domain, where G is the Fourier Series of g , and thereby to a similar relationship for the VTFs, where the fast decay of the Fourier coefficients can be used to characterise the closeness (Theorem 2). The VTF result then allows the closeness within the DoA and the convergence speed to be easily established (Theorem 3). Finally we also provide direct closeness results for the ACVF (Theorem 4).

3.1 Closeness of the Spectrum

We are ultimately interested in characterising the closeness of the covariance structure of a fractionally differenced process to that of its fGn fixed point at large lags. The rate of decay of the sequence of Fourier coefficients of a function is well known to be closely connected to its smoothness properties. It is, therefore, unsurprising that a notion of closeness in the spectral domain can take the form of statements about smoothness of the function g in (6).

The following spectral closeness result is the crucial basis for both the VTF and ACVF results to come.

Theorem 1. Assume that $H \in [1/2, 1)$ and define $g(x) := f_H(x)/f_H^*(x)$, $x \neq 0$ and $g(0) := \lim_{x \rightarrow 0} g(x)$. Then $g(0) = c_f/c_f^* = h(0)/(2\pi VC(H))$ and g satisfies the following over $[-1/2, 1/2]$:

- (i) g is even, continuous, positive, bounded, and L^p , $p > 0$;
- (ii) g is twice differentiable, and smooth away from $x = 0$;
- (iii) $g'' \in \Lambda_{2H-1} \cap V$, but $g'' \notin \Lambda_{\beta'}$ for $\beta' > 2H - 1$;
- (iv) g admits a Fourier series with coefficients $\{G_j\}$ such that $\sum_{j=-\infty}^{\infty} j^2 |G_j| < \infty$ and $G_n = O(n^{-3})$. In particular $\sum_{j=-\infty}^{\infty} |G_j| < \infty$ and $\sum_{j=-\infty}^{\infty} j^\alpha |G_j| < \infty$ for $1 < \alpha < 2$.

Proof. Unless otherwise specified, we consider the domain $x \in [-1/2, 1/2]$.

First, since $f_H(x) \stackrel{x \rightarrow 0}{\sim} c_f^* |x|^{-(2H-1)}$ and $f_H^*(x) \stackrel{x \rightarrow 0}{\sim} c_f |x|^{-(2H-1)}$, $g(0) := \lim_{x \rightarrow 0} g(x) = c_f/c_f^*$.

The proof of (i) is straightforward. For completeness, details are provided in the appendix.

To prove the smoothness properties (ii) and (iii), we first establish those of \tilde{g} defined as

$$\tilde{g}(x) := \frac{c_f \pi^{2H+1}}{c_f^* h(0)} \cdot \frac{h(x)}{g(x)} \quad (7)$$

$$= \left| \frac{\sin(\pi x)}{\pi x} \right|^{2H+1} + |\sin(\pi x)|^{2H+1} \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} |\pi j + \pi x|^{-(2H+1)} \quad (8)$$

$$:= |a(x)|^{2H+1} + |b(x)|^{2H+1} c(x). \quad (9)$$

It is not difficult to show (see the appendix for details) that \tilde{g} is smooth everywhere except at the origin where its smoothness is controlled by that of $|b|^{2H+1}$, which we now study.

Let $\beta = 2H - 1$. Since b is smooth and $\beta \in (0, 1)$, $|b|^{\beta+2}$ is twice differentiable at the origin. The smoothness of its second derivative is controlled by $(b')^2 |b|^\beta$, which, since $b \in \Lambda_1$ and $x \mapsto |x|^\beta$ is in Λ_β , is also in Λ_β by the multiplicative and compositional closure properties of Λ_β . It follows that \tilde{g}'' exists and is in Λ_β . Since however $x \mapsto |x|^\beta$ is not in $\Lambda_{\beta'}$ for any $\beta' > \beta$, and moreover $b(x) \stackrel{x \rightarrow 0}{\sim} \pi x$ and $b'(0) \neq 0$, \tilde{g}'' is not in $\Lambda_{\beta'}$ for any $\beta' > \beta$.

Since smooth functions are in V , by similar arguments using the closure properties of V , we have $\tilde{g}'' \in V$ if $|b|^\beta \in V$. The latter holds since it is easy to see that $|b|^\beta$ is monotone (with total variation 2).

We have shown that \tilde{g}'' exists and is in $\Lambda_{2H-1} \cap V$, but not in $\Lambda_{\beta'}$ for any $\beta' > 2H - 1$. We now prove the same for g using (7). It suffices to consider $1/\tilde{g}$ since h''' exists. Since \tilde{g} is bounded away from zero, (ii) follows since $(1/\tilde{g})'' = 2(\tilde{g}')^2/\tilde{g}^3 - \tilde{g}''/\tilde{g}^2$ clearly exists, and is smooth away from the origin. Now consider (iii). It follows from the last expression and the fact that $\tilde{g} > 0$ that $(1/\tilde{g})''$ and hence g'' are in V and Λ_{2H-1} by applying the respective closure properties. Finally, since $1/\tilde{g}^2(0) \neq 0$, the smoothness of $(1/\tilde{g})''$ is controlled by that of \tilde{g}'' and so $(1/\tilde{g})'' \notin \Lambda_{\beta'}$ for any $\beta' > 2H - 1$. This completes the proof of (iii).

We now prove (iv). Since each of g , g' , and g'' are continuous and bounded, the Fourier series for each exists and are related by term by term differentiation (Champeney (1990), Thm. 15.19). In particular $g(x) = \sum_{j=-\infty}^{\infty} G_j e^{2\pi i j x}$, and we can write $g''(x) = -4\pi^2 \sum_{j=-\infty}^{\infty} j^2 G_j e^{2\pi i j x}$. Now Zygmund Zygmund (2002), Thm. VI.3.6 states that the Fourier Series of a function in $\Lambda_\beta \cap V$ for some $\beta > 0$ converges absolutely. This applies to g'' and proves that $\sum_{j=-\infty}^{\infty} j^2 |G_j| < \infty$ as claimed. Finally, since $g'' \in V$, the magnitude of its Fourier coefficients decay as $O(|j|^{-1})$ (Zygmund Zygmund (2002), Thm. II.4.12), proving that $G_j = O(j^{-3})$. \square

The result suggests that fractionally differenced processes are not typical; for a general LRD process, only boundedness of g at the origin would be automatic. In contrast, the present g is a very well behaved function. A plot of g is provided in Figure 1 which shows its flatness at the origin (it also suggests that g is monotone increasing over $[0, 1/2]$, though this plays no role in what follows). Here we have set $c_f = c_f^*$, so that its value at the origin is just 1. It is interesting to note that since g is positive, even, and square integrable, it is the spectral density of some second order time series.

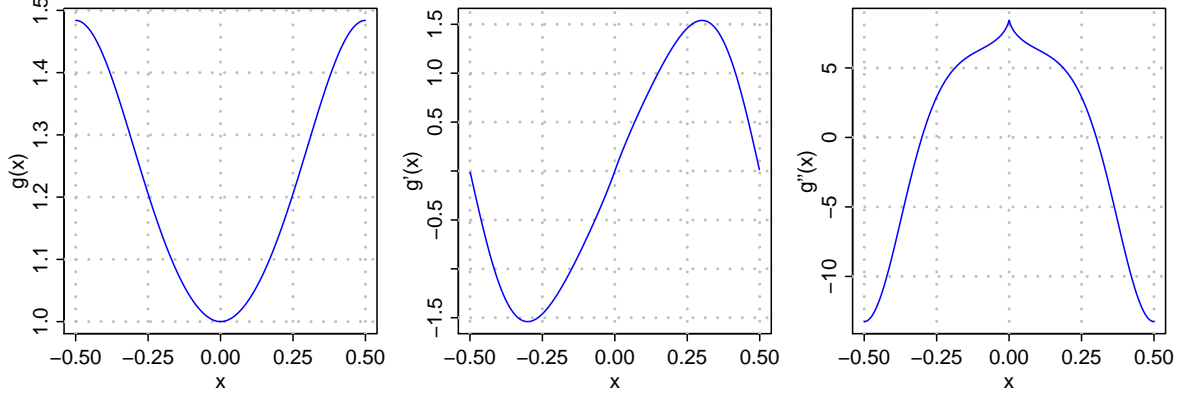


Figure 1: The function $g(x) = f_H(x)/f_H^*(x)$ and its first two derivatives in the canonical case of a pure fractionally differenced process (FARIMA0d0) with $H = 0.8$ and $c_f = c_f^*$.

3.2 Closeness of the VTF

The first step in elucidating the relationship between ω_H and ω_H^* is to confirm that the relationship $f_H(x) = f_H^*(x)g(x)$ between the spectral densities translates to the expected convolution relationship $\gamma_H = \gamma_H^* \star G$ between the ACVFs. It is straightforward to confirm that, thanks to the nice behaviour of g and G detailed in Theorem 1, this is indeed the case.

Lemma 1. *The auto-covariance functions γ_H and γ_H^* are related through the convolution $\gamma_H = \gamma_H^* \star G$.*

For completeness a proof is given in the appendix.

Since $\omega_H = \mathbf{I}\gamma_H$, it is tempting to seek a relationship of the form $\omega_H = G \star \omega_H^*$ through taking the ‘double integral’ of $\gamma_H = G \star \gamma_H^*$. However, since $\omega_H^*(m) = \mathcal{V}m^{2H}$ diverges with m , this is not necessarily well defined. The following lemma provides a sufficient condition for the existence of such a convolution, as well as some of its important properties which will be crucial in what follows.

Lemma 2. *Assume $1 < \alpha < 2$ and let $a = \{|n|^\alpha, n \in \mathbf{Z}\}$. Let b be a symmetric sequence satisfying $\sum_{j=1}^{\infty} j^\alpha |b_j| < \infty$. Then $S_b = \sum_{j=-\infty}^{\infty} b_j$ and the symmetric sequence $c = a \star b$ exist, and $(c_n - S_b a_n) \xrightarrow{n \rightarrow \infty} 0$.*

This result is proved in the appendix. The proof of the last part is based on the monotonicity of a function which generalises γ_H^* to two parameters (see Lemma A1 in the appendix).

Corollary 1. *The convolution $G \star \omega_H^*$ exists for $H \in (1/2, 1)$.*

Proof. Set $b = G$ in Lemma 2. The condition on b holds since $\sum_{j=1}^{\infty} j^\alpha |G_j| < \sum_{j=1}^{\infty} j^2 |G_j|$ which is finite from Theorem 1. The result then follows immediately by identifying α with $2H$ and a with ω_H^* . \square

The following lemma shows that, if existence is granted, taking the ‘double integral’ of a convolution is straightforward, provided a double counting issue at the origin is allowed for.

Lemma 3. *Let a, b be symmetric sequences and assume that $c := a \star b$ exists. Then $\mathbf{I}c$ exists, and if $(\mathbf{I}a) \star b$ exists, then $\mathbf{I}c = (\mathbf{I}a) \star b - ((\mathbf{I}a) \star b)_0$.*

The proof of this result is based on a careful rearrangement of terms justified by the repeated use of the existence of $(\mathbf{I}a) \star b$. It is given in the appendix.

We are now able to prove our main result on the VTF.

Theorem 2. *Let ω_H denote the VTF of a fractionally differenced process with $H \in (1/2, 1)$ and c_f chosen equal to c_f^* . Then*

$$\omega_H(n) = \omega_H^*(n) + D + o(1)$$

where $D = -2 \sum_{j=1}^{\infty} j^\alpha |G_j| < 0$ is a constant.

Proof. Since each of $\gamma_H^* \star G$ and $\omega_H^* \star G$ exist, Lemma 3 applies upon identifying $a = \gamma_H^*$, $b = G$ and $c = \gamma_H$ and states that $\omega_H = \omega_H^* \star G - \{\omega_H^* \star G\}(0)$. From Lemma 2 with $b = G$, $S_G = \sum_{j=-\infty}^{\infty} G_j < \infty$ exists. By introducing the term $S_G \omega_H^*$ we obtain

$$\begin{aligned} \omega_H &= S_G \omega_H^* + \left(\omega_H^* \star G - S_G \omega_H^* \right) - \{\omega_H^* \star G\}(0) \\ &= S_G \omega_H^* + o(1) - 2 \sum_{j=1}^{\infty} j^\alpha |G_j| \end{aligned}$$

by the final part of Lemma 2. Since $S_G = g(0) = c_f/c_f^* = 1$, the result follows. \square

The key property underlying this result is $\omega_H^* \star G - S_G \omega_H^* \xrightarrow{n \rightarrow \infty} 0$, which shows that G is ‘compact’ enough to act as an aggregate multiplier S_G asymptotically. This is analogous to the role the covariance sum $S_\gamma := \sum_{k=-\infty}^{\infty} \gamma(k)$ plays in the asymptotic variance of aggregated short-range dependence processes Gefferth et al. (2003).

3.3 Atypicality and Speed of Convergence

Theorem 2 showed that the VTF of a fractionally differenced process is asymptotically equal to the VTF of its fGn fixed point up to an additive constant. This makes fractionally differenced process highly atypical among LRD processes. We show this first for the VTF itself, and then for the speed of convergence of the CTF to the fixed point.

Without loss of generality, the VTF of any time series in the domain of attraction of a given fGn can be expressed as

$$\omega_H(n) = \omega_H^*(n) + \omega_d(n) \tag{10}$$

where ω_d represents the distance of the VTF from its limiting fGn counterpart. By definition, $\omega_d(n) = o(n^{2H})$, but otherwise the growth rate of ω_d is not constrained, implying that there is considerable variety within the domain of attraction.

One way of characterising the size of the difference $\omega_d(n)$ is to use *regular variation* Bingham et al. (1987); Gefferth et al. (2003). A regularly varying function $f(n)$ of index β and integer argument $n \in \mathbf{N}^+$ satisfies $\lim_{k \rightarrow \infty} f(kn)/f(k) = n^\beta$, $\beta \in \mathbf{R}$. Assume without loss of generality that ω_d is upper bounded by a regularly varying function of index $\beta \in [0, 2H]$, that is

$$\omega_d(n) = O(s(n)n^\beta), \tag{11}$$

where s is a *slowly varying* function (that is regularly varying with index 0), and β is the infimum of indices for which (11) holds. A notion of *closeness* of the process to the limiting fGn can then be defined in terms of β , where the smaller the index, the closer the process.

According to this scheme, Theorem 2 states that fractionally differenced processes belong in the closest layer of the hierarchy, corresponding to $\beta = 0$. Furthermore, the theorem shows that $s(n)$ (which could in general

diverge, for example $s(n)^{n \rightarrow \infty} \log(n)$ tends to a constant. Thus, the VTF of a fractionally differenced process lies in a very tight neighbourhood indeed of the VTF of its limiting fixed point. Far from being typical LRD processes, they deviate only in very subtle ways from fGn in terms of their large lag behaviour.

>From (2), there is a direct relationship between closeness in the above sense and speed of convergence of the CTF to its fixed point under aggregation.

Theorem 3. *Let ϕ_H denote the CTF of a fractionally differenced process in the domain of attraction of ϕ_H^* with $H \in (1/2, 1)$. Then*

$$\phi_H^{(m)}(n) = \phi_H^*(n) + D(1 - n^{2H})m^{-2H} + o(m^{-2H}) = \phi_H^*(n) + O(m^{-2H})$$

where D is the constant from Theorem 2.

Proof. The result follows from substituting $\omega_H(n) = \omega_H^*(n) + D + o(1)$ from Theorem 2 in (2) and using $(1+x)^{-1} = 1 - x + O(x^2)$. \square

Beginning from (10), it holds generally for LRD processes in the DoA of ϕ_H^* that $\phi_H^{(m)}(n) = \phi_H^*(n) + O(s(m)m^{-2H+\beta})$. It follows that fractionally differenced processes, for which $\beta = 0$ and $s(m)$ is identically equal to a constant, converge faster to the fixed point compared to all other processes in the DoA. Examples are provided in Section 5.

4 Closeness of the ACVF

Recall that $\omega = \mathbf{I}\gamma$. Because the double sum operator \mathbf{I} smooths out local variations, Theorem 2 can not be used to derive an explicit characterisation of the closeness in terms of the ACVF. We therefore set out to provide a closeness result for the ACVF here. Not only is this of interest in its own right, it also provides an alternative way of demonstrating the closeness to fGn, as well as leading to an additional result on the spectral closeness to fGn in an additive sense.

The following lemma is the analogue of Lemma 2 used for the ACVF. A proof is given in the appendix.

Lemma 4. *Assume $-1 \leq \alpha < 0$ and let a be the symmetric positive sequence $a_n = |n|^\alpha$, $n \neq 0$ and $a_0 > 0$. Let b be a symmetric sequence with $|b_0| < \infty$ for which there exists $\beta \in [0, 2]$ such that $\sum_{j=1}^{\infty} j^\beta |b_j| < \infty$ and $|b_n| = O(n^{-(\beta+1)})$. Then $S_b := \sum_{j=-\infty}^{\infty} b_j$ and the symmetric sequence $c := a \star b$ exist, and $c_n - S_b a_n = O(n^{\alpha-\beta})$ as $n \rightarrow \infty$.*

We can now prove the ACVF closeness result

Theorem 4. *Let γ_H denote the ACVF of a fractionally differenced process with $H \in (1/2, 1)$ and c_f chosen equal to c_f^* . Then*

$$\gamma_H(n) = \gamma_H^*(n) + O(n^{2H-4})$$

Proof. The exact ACVF of a unit variance fGn(H) is given by

$$\gamma_H^*(n) = \frac{1}{2}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}),$$

for $n \geq 0$ and $\gamma_H^*(n) = \gamma_H^*(-n)$ for $n < 0$. Then $\gamma_H^*(0) = 1$, and for $n \neq 0$ $\gamma_H^*(n) = (1/2)|n|^{2H}k(|n|^{-1})$ where $k(x) := (1+x)^{2H} + (1-x)^{2H} - 2$. Expanding k in a Taylor series around the origin, we obtain the following series representation:

$$\gamma_H^*(n) = \sum_{j=1}^{\infty} c_j f_j(n), \quad c_j := \frac{\prod_{i=0}^{2j-1} (2H-i)}{(2j)!}, \quad f_j(n) := \begin{cases} |n|^{2H-2j} & n \neq 0 \\ \mathbf{1}\{j=1\}/c_1 & n = 0 \end{cases}$$

which is uniformly absolutely convergent since $\{a_j\}$ is absolutely convergent by the ratio test.

Now $\gamma_H(n) = (\gamma_H^* \star G)(n) = \sum_{k=-\infty}^{\infty} G(k) \sum_{j=1}^{\infty} c_j f_j(n-k) = \sum_{j=1}^{\infty} c_j (f_j \star G)(n)$ where the existence of $\gamma_H^* \star G$ and γ_H^* as absolutely convergent series justifies the interchange of summations (Apostol (1974), Thm. 8.43). We can now compare γ_H and γ_H^* as

$$|\gamma_H(n) - \gamma_H^*(n)| = \sum_{j=1}^{\infty} |c_j| |(f_j \star G)(n) - f_j(n)| \quad (12)$$

$$\leq |c_1| |(f_1 \star G)(n) - f_1(n)| + \sum_{j=2}^{\infty} |c_j| |(f_j \star G)(n)| + O(n^{2H-4}); \quad (13)$$

We shall show that each of the terms on the right hand side are of order $O(n^{2H-4})$.

The result for the first term follows immediately from Lemma 4 upon identifying f_1 with a , $2H-2$ with α , G with b with a choice of $\beta = 2$ (justified by Theorem 1(iii)), and noting that $\sum_{j=-\infty}^{\infty} G_j = 1$ by the assumption $c_f = c_f^*$.

Now consider the second term. Recall from Theorem 1 that $G_n = O(n^{-3})$, i.e. there exists $K > 0$ such that $G_n \leq K|n|^{-3}$ for n sufficiently large. Thus, when $j \geq 2$ and for $n > 0$ large enough

$$\begin{aligned} |(f_j \star G)(n)| &= \sum_{k=-\infty}^{\infty} |f_j(k)| |G_{n-k}| = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} |k|^{2H-2j} |G_{n-k}| \leq \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} |k|^{2H-4} |G_{n-k}| \\ &= \sum_{k=1}^{\infty} |k|^{2H-4} |G_{n+k}| + \sum_{k=1}^{\lfloor n/2 \rfloor} |k|^{2H-4} |G_{n-k}| + \sum_{k=\lfloor n/2 \rfloor + 1}^{\infty} |k|^{2H-4} |G_{n-k}| \\ &\leq K \sum_{k=1}^{\infty} |k|^{2H-4} (n+k)^{-3} + K \sum_{k=1}^{\lfloor n/2 \rfloor} |k|^{2H-4} (n-k)^{-3} + \left| \frac{n}{2} \right|^{2H-4} \sum_{k=\lfloor n/2 \rfloor + 1}^{\infty} |G_{n-k}| \\ &\leq K \sum_{k=1}^{\infty} (kn + k^2)^{2H-4} + K \sum_{k=1}^{\lfloor n/2 \rfloor} (kn - k^2)^{2H-4} + \left| \frac{n}{2} \right|^{2H-4} \sum_{k=-\infty}^{\infty} |G_{n-k}| \\ &< K \sum_{k=1}^{\infty} (kn)^{2H-4} + K \sum_{k=1}^{\lfloor n/2 \rfloor} (kn/2)^{2H-4} + \left| \frac{n}{2} \right|^{2H-4} \sum_{k=-\infty}^{\infty} |G_k| \\ &= O(n^{2H-4}); \end{aligned}$$

using $2H-4 \geq -3$, that $kn + k^2 \geq kn$ for all k , $kn - k^2 \geq nk/2$ for $1 \leq k \leq n/2$, the absolute summability of G , and the fact that $\sum_{k=1}^{\infty} |k|^{2H-4} < \infty$. Hence the right hand side of (12) is $O(n^{2H-4})$. \square

In Section 3.1 we derived a result which may best be described as ‘multiplicative closeness’ for the spectrum of a fractionally differenced process. This form of closeness was natural for providing a subsequent link to the time domain. However, when calculations with the frequency domain are of specific interest, an additive closeness result for the spectrum is useful. Such a result can easily be derived from the above theorem.

Corollary 2. *It holds that $f_H(x) = f_H^*(x) + \varphi(x)$ where φ is differentiable, $\varphi' \in \Lambda_\alpha$ if $\alpha < 2 - 2H$, and $\varphi(0) = 0$. Moreover, $\varphi(x) = O(x^{-2H+3})$ as $x \rightarrow 0$.*

Proof. Let $\varphi := f_H - f_H^*$. The Fourier series of φ exists and equals φ , and its coefficients are given by $d_n = \gamma_H(n) - \gamma_H^*(n)$, which by Theorem 4 is $O(|n|^{2H-4})$. Since $2H-4 < -2$ the first absolute moment of the coefficients exists, so Theorem 7.19 Kufner and Kadlec (1971), applies and shows that φ' exists and $\varphi' \in \Lambda_{2-2H}$. By the definition of $g(0)$, $\varphi(0) = \lim_{x \rightarrow 0} (f_H(x) - f_H^*(x)/f_H^*(x)) = 0$.

The last claim follows by straightforward expansion of $f_H(x) - f_H^*(x)$ about $x = 0$. Details are given in the appendix. \square

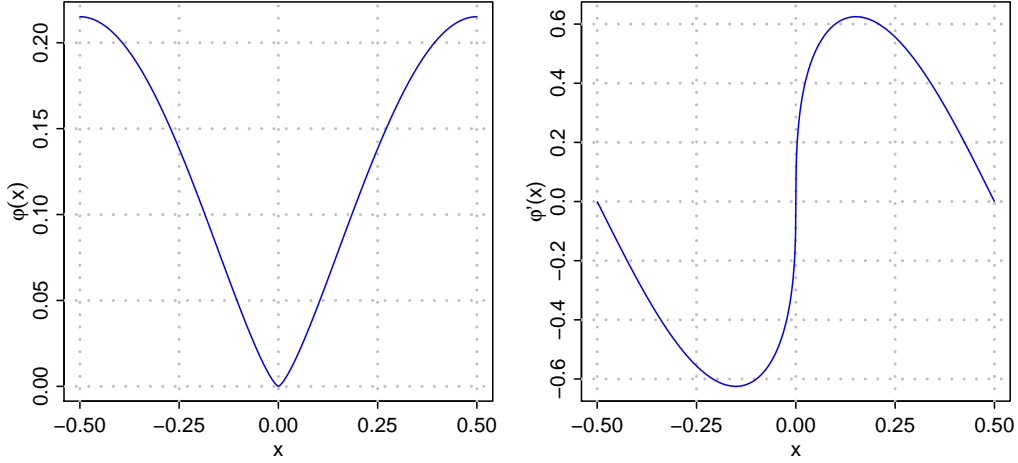


Figure 2: The function $\varphi(x) = f_H(x) - f_H^*(x)$ and its first derivative in the canonical case of a pure fractionally differenced process (FARIMA0d0) with $H = 0.8$ and $c_f = c_f^*$.

The additive closeness of the spectrum is a highly non-trivial result: from the usual spectrum definition of LRD (Section 2.1), LRD with Hurst parameter H implies only that the ratio between f_H/f_H^* is bounded at the origin whereas the difference $f_H - f_H^*$ generally diverges. That the difference is not only a bounded function but tends to zero, and is also differentiable, emphasizes in yet another way how unusual fractionally differenced processes are among LRD processes. To explore this in more detail, observe that the statement of Corollary 2 can be written

$$f_H + \varphi^- = f_H^* + \varphi^+$$

where $\varphi^- \geq 0$ and $\varphi^+ \geq 0$. Both φ^+ and φ^- define spectral densities with $\varphi^+(0) = \varphi^-(0) = 0$. We then (Brockwell and Davis (1991), Cor. 4.3.1) obtain a probabilistic variant of the closeness result: a fractionally differenced process is equal in the distributional sense to its limiting fGn up to additive independent processes with spectra φ^+, φ^- , both of which have the property of having a vanishing covariance sum $S_\gamma = \sum_{j=-\infty}^{\infty} \gamma_j$. Such processes (called *Constrained Short Range Dependent* (CSR D) in Gefferth et al. (2003)), lie in the DoA of an fGn with Hurst parameter $H' \in [0, 1/2)$. In contrast, for Short Range Dependent (SRD) processes (those in the DoA of a fGn with $H' = 1/2$), S_γ is finite but positive. A graph of a particular φ and its first derivative is shown in Figure 2. The plot suggests that $\varphi^- \equiv 0$; whereby FARIMA would be equal in distribution to fGn plus an independent CSR D process.

To conclude our treatment of the ACVF, observe that a slightly weaker form of the closeness result of Theorem 2 can be derived from Theorem 4. Indeed, the identity $\omega_H(n) - \omega_H^*(n) = (\mathbf{Id})_n$ implies

$$|\omega_H(n) - \omega_H^*(n)| = \left| \sum_{k=0}^{n-1} \left(\sum_{j=-\infty}^{\infty} d_j - \sum_{j=-k}^k d_j \right) \right| \leq 2 \sum_{k=0}^{n-1} \sum_{j=k+1}^{\infty} |d_j| \leq O(1) \sum_{k=0}^{n-1} k^{2H-3} = O(1),$$

where we have used that $d_n = O(|n|^{2H-4})$ implies $\sum_{j=k+1}^{\infty} |d_j| = O(1) \sum_{j=k+1}^{\infty} j^{2H-4} = O(k^{2H-3})$. The $O(1)$ remainder term simply corresponds to a bounded function; this is clearly somewhat weaker than the asymptotically constant remainder term appearing in Theorem 2.

We recently became aware of Lieberman & Phillips (2008) Lieberman and Phillips (2008) which provides an asymptotic expansion for a class of fractionally differenced processes corresponding to (3), though $h(x)$ is required to be smooth rather than C^3 . Using the first two terms of this expansion and comparing with an expansion for $\gamma_H^*(m)$, it is possible to recover the $O(n^{2H-4})$ term of Theorem 4. The work of Lieberman and Phillips (2008) is focussed on numerical approximation through infinite-order asymptotic expansions and does not compare against fGn or draw conclusions on convergence speed or brittleness as we do here.

5 Fractional Processes are Brittle

As pointed out at the end of Section 3, fractionally differenced processes converge ‘almost immediately’ to their fGn fixed point compared to other processes in the domain of attraction, and this is true in terms of each of the VTF, ACVF and spectrum. In this section we point out and illustrate a key consequence of this fact, namely the *brittleness* of fractionally differenced models.

5.1 Brittleness

Experimental data, especially data measured on a continuous scale, is very rarely clean. Imperfections in physical measurement are often treated through the concept of observation noise, modelled as a random process which perturbs the underlying observables. A very common choice is that of additive independent Gaussian noise, either white or coloured. In the present context, this corresponds to adding to the original VTF (or ACVF, or spectrum) the VTF (respectively ACVF, spectrum) of a short range dependent noise process, that is a noise whose own fGn fixed point has $H' = 1/2$.

As argued at the end of the previous section, we can essentially think of a fractionally differenced process as an fGn to which a CSRD process has been added. Adding an SRD noise to this will change the asymptotic behaviour, because the SRD asymptotics (with $S_\gamma > 0$) is ‘stronger’ than CSRD asymptotics (with $S_\gamma = 0$). In terms of the hierarchy within the DoA described by the index β from (11), whereas the original process lies very close to the centre with $\beta = 0$, the SRD-perturbed process will lie considerably further out, with $\beta = 1$. A similar observation can be made if we instead add a noise with LRD with $H' < H$ (resulting in $\beta \in (1, 2H)$), or even another CSRD process with $H' > 0$ (resulting in $\beta \in (0, 1)$). This last result follows from the fact that Theorem 2 implies that the ‘error’ processes are so special that they are not only CSRD, but correspond to the extreme case of $H' = 0$, resulting in $\beta = 0$.

Since the addition of even trace amounts of noise of diverse kinds will change the asymptotics, pushing the process further from its fGn limit and therefore slowing its convergence rate to it under aggregation, fractional differencing models are ‘brittle’ or non-robust in this sense. Properties of systems driven by such processes may therefore differ qualitatively from properties of the same system once noise is added. The precise impact of the noise is beyond the scope of this paper (see the discussion). It will depend on both the application and the class of noise and must be determined case by case.

5.2 Numerical Illustrations

In this section we illustrate the brittle nature of fractionally differenced processes through high accuracy numerical evaluation of the VTF of FARIMA time series, both with and without additive noise.

Three different examples will be considered, two with SRD-noise and one with LRD-noise. More precisely, the perturbed processes are $Z_i(t) = X_i(t) + \sqrt{0.1}Y_i(t)$ for $i = 1, 2, 3$, where

- 1) X_1 : unit variance FARIMA(0, 0.3, 0);
 Y_1 : unit variance Gaussian white noise,
- 2) X_2 : unit variance FARIMA(1, 0.3, 1) with ARMA parameters $(\phi_1, \theta_1) = (0.3, 0.7)$;
 Y_2 : unit variance ARMA(1, 1) process also with ARMA parameters $(\phi_1, \theta_1) = (0.3, 0.7)$,
- 3) X_3 : unit variance FARIMA(0, 0.3, 0);
 Y_3 : unit variance FARIMA(0, 0.2, 0).

In each case, the original process X_i and the perturbed process Z_i share a common fGn fixed point, but have unequal variances. It may seem unfair to compare results for processes with different variances, however the opposite is true. In fact, if the variances of Z_i and X_i were chosen equal, this would mean that $c_f \neq c_f^*$, and so their fGn limits would be different, rendering meaningful comparison impossible. To see this more directly, from the definitions in Section 2.1 it is clear that adding a perturbation corresponding to a smaller H value does not alter the fixed point. On the other hand the variance must increase when an independent noise is added.

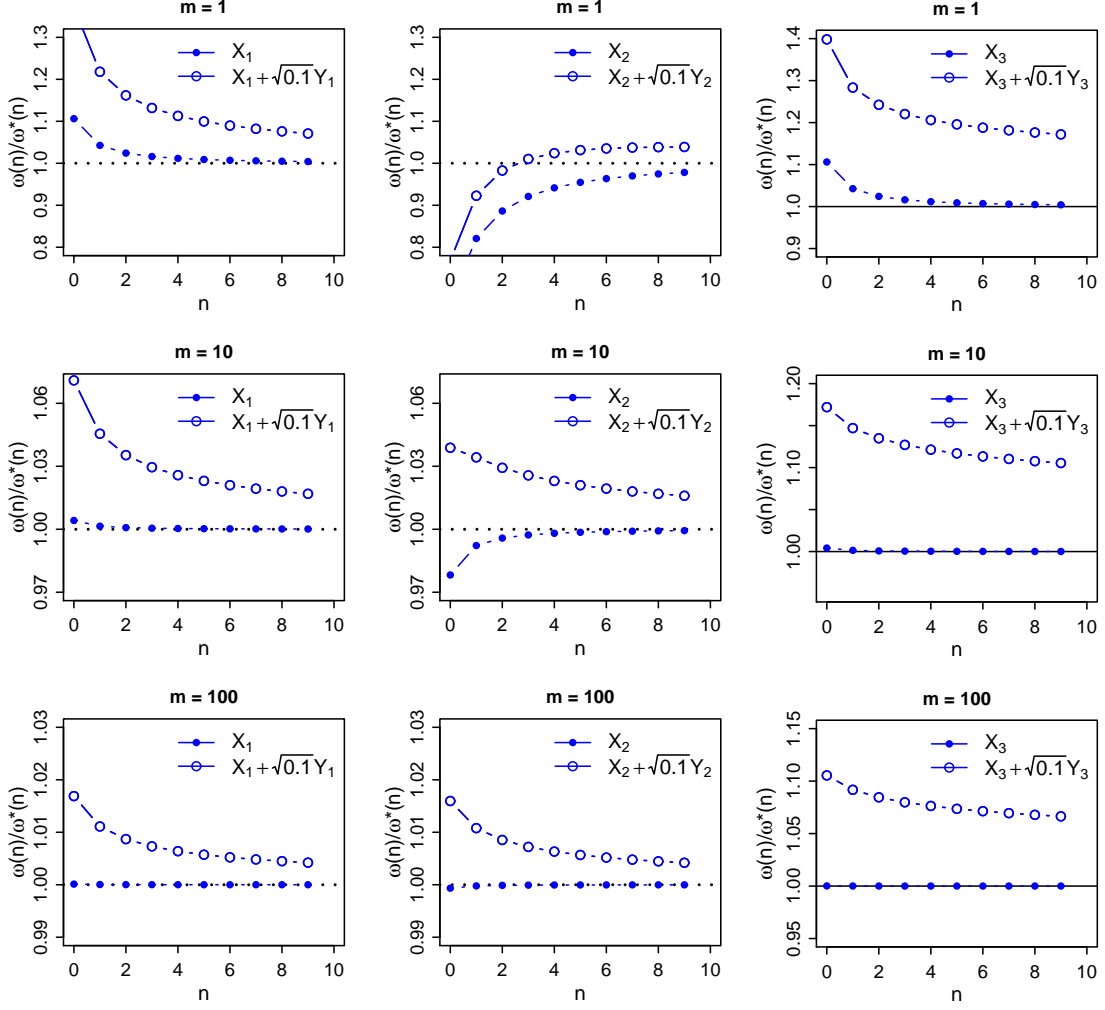


Figure 3: Ratios of VTF's of original FARIMA and perturbed processes to their fGn limit, both originally and under aggregation, one column per example. The solid circles denote unperturbed FARIMA; the hollow circles the perturbed ones. It is seen that the VTF's for unperturbed FARIMA converge much faster than their perturbed counterparts.

For each example $i = 1, 2, 3$, we calculate the VTF of Z_i and X_i and normalise them by dividing by their common fGn limit ω_H^* . Closeness to fGn can therefore be evaluated by looking to see how the normalised VTF deviates from 1 for each lag. Maple version 13 was used to numerically evaluate the variance time functions to a high degree of precision.

Figure 3 displays the normalised VTFs for lags 1-10 for aggregation levels $m = 1, 10$, and 100 , with one example per column. The graphs clearly demonstrate that even a small departure from FARIMA takes the process much further away from its corresponding fGn. Indeed, after an aggregation of level 100 , in each case the VTF of the original process is visually indistinguishable from its fGn limits compared to their perturbed versions.

Note that the second column in the figure gives an example where before aggregation ($m = 1$) the perturbed process was in fact *closer* to the fixed point over the first few lags, where most of the obvious autocovariance lies. Under aggregation however, this quickly reverses as the different asymptotic behaviours of the original and perturbed processes manifest and become dominant at all lags.

6 Discussion

We have shown that fractionally differenced processes have an asymptotic autocovariance structure which is extremely close to that of the fractional Gaussian noise, more specifically, to that of the fGn fixed point to which the given process will tend under aggregation based renormalisation. We have shown this independently for each of three equivalent views of the autocovariance structure, namely behaviour of the spectral density at the origin, and each of the autocovariance function and the variance time function in the large lag limit.

We showed that the natural class of processes against which this behaviour should be compared are those in the domain of attraction of the fGn fixed point limit. Using regular variation to provide a measure of distance from this fixed point within the DoA, we were able to precisely quantify the nature of this ‘closeness’, and to confirm that the fractionally differenced class are indeed exceptionally unusual in this regard, resulting in very fast convergence to fGn under renormalisation. We then used this fact to point out that the fractionally differenced process class is brittle, that is, non robust to the presence of noise. In particular we showed that the addition of arbitrarily small amounts of independent noise, not only Gaussian white noise but also noises which are much gentler in a precise sense, changes the asymptotic covariance structure qualitatively. This fact has not been appreciated in the literature where such models, for example the FARIMA class, are widely used in time series modelling, synthetic data generation, and to drive more complex stochastic systems such as queuing systems, without regard to robustness with respect to the model in this sense.

The assessment of the impact of the brittleness of fractionally differenced models is beyond the scope of this work, as it will depend intimately on each particular application as well as the nature of the noise in question. However, we argue that conclusions based on the perception that FARIMA and related models represent ‘typical’ LRD behaviour need to be reassessed, in particular in contexts where noise is important to consider. To give an example of a possible impact in the noiseless case, we conclude by expanding upon the comments given in the introduction on statistical estimation.

The closeness of a process to its fGn fixed point in functional terms is directly related to the speed of convergence of that process to the fixed point under aggregation. One application where this fact carries direct implications is the performance of statistical estimators for the Hurst parameter H . Fundamentally, semi-parametric estimators of scaling parameters such as H are based on underlying estimates made at a set of ‘aggregations’ at different levels, that is at multiple scales Robinson (1994); Beran (1994); Abry et al. (1998); Taqqu et al. (1995). The sophistication of particular estimators notwithstanding, this is true regardless of whether they are based in the spectral, time, or wavelet domains, though the technical details vary considerably. In the time domain using time domain aggregation the link is of course direct, and reduces to looking at the asymptotically power-law nature of $\mathcal{V}^{(m)} = \omega^{(m)}(0)$ as a function of m in some form. This is precisely where fractionally differenced processes are at a real advantage, as this quantity converges extremely quickly to that of the fGn fixed point, whose ideal power-law behaviour $\mathcal{V}^{(m)} = \mathcal{V}m^{2H}$ allows H to be easily recovered. As a result, estimator performance evaluated through the use of fractionally differenced models would be superior to that for LRD processes more generally. Note that we are not recommending that H estimation be performed directly in the time domain by regressing $\hat{\mathcal{V}}^{(m)}$ on m , indeed we have argued the opposite Abry et al. (1998). Our point is that the extreme closeness of such models to fGn must ultimately manifest in simpler asymptotic behaviour which will, in general, translate to improved estimation. Indeed, in the spectral domain, the importance of the degree of smoothness at the origin for the ultimate limits on estimator performance has already been noted Giraitis et al. (1997). Note that the above observations in no way put into question the findings of prior work on estimation of fractional processes in noise.

7 Appendix

The appendix is split according to results relating to spectral closeness (Section 3.1), closeness of the VTF (Section 3.2), and of the ACVF (Section 4). For convenience, the statement of results proved here are generally repeated. Lemmas A1 and A2 are labelled separately as they appear in the appendix only.

7.1 Spectrum

Details of the proof of Theorem 1

(i) It is well known, and can be verified by examining (4) and (5), that each of $f_H^*(x)$ and $f_H(x)$ diverge to infinity at $x = 0$ but are otherwise even, positive and continuous. Since $g(0) > 0$ is finite, g is positive and continuous on the compact domain and hence bounded, and even. Since g is continuous, it is integrable (Champeney (1990), p.9), and since, for $p > 0$, g^p is likewise continuous, positive and bounded, g is in L^p .

(ii) Since $a(x) > 0$, \tilde{g} is bounded away from zero. Each of a , b , and c are smooth. The latter follows from the fact that for each $j \neq 0$ the term $|\pi j + \pi x|^{-(2H+1)}$ is infinitely differentiable in $x \in [-1/2, 1/2]$. By comparing against $\sum_{j=1}^{\infty} (j-1/2)^{-(2H+1)} < \infty$ the Weierstrass' M -test shows that the defining sum for c , and the sum of the term by term first derivatives, each converge uniformly. A classical result on the differentiability of infinite series (Apostol (1974), Thm. 9.14) then shows that c' is given by the latter sum. Using exactly the same M -test, this can be repeated for derivatives of all orders, proving that c is smooth.

Since a is smooth and bounded above zero, $|a|^{2H+1}$ is smooth over $[-1/2, 1/2]$, and the same is true for $|b|^{2H+1}$ away from the origin. It follows that \tilde{g} is smooth everywhere except at the origin where its smoothness is controlled by that of $|b|^{2H+1}$.

7.2 VTF

Lemma 1. *The auto-covariance functions γ_H and γ_H^* are related through the convolution $\gamma_H = G \star \gamma_H^*$.*

Proof. The r.h.s. exists since $\sum_{j=-\infty}^{\infty} G_j \gamma_H^*(n-j) \leq \sum_{j=-\infty}^{\infty} |G_j| |\gamma_H^*(n-j)| \leq \gamma_H^*(0) \sum_{j=-\infty}^{\infty} |G_j| < \infty$ from Theorem 1. For the l.h.s. we can write

$$\gamma_H(n) = \int_{-1/2}^{1/2} f_H(x) e^{2\pi i x n} dx = \int_{-1/2}^{1/2} g(x) f_H^*(x) e^{2\pi i x n} dx \quad (14)$$

$$= \int_{-1/2}^{1/2} \left(\sum_{j=-\infty}^{\infty} G_j e^{-2\pi i x j} \right) f_H^*(x) e^{2\pi i x n} dx \quad (15)$$

as the Fourier series for $g(x)$ converges absolutely for all x since $\sum_{j=-\infty}^{\infty} |G_j| < \infty$ (Theorem 1). Now

$$\begin{aligned} \int_{-1/2}^{1/2} \left(\sum_{j=-\infty}^{\infty} |G_j e^{-2\pi i x j}| \right) f_H^*(x) e^{2\pi i x n} dx &= \sum_{j=-\infty}^{\infty} |G_j| \int_{-1/2}^{1/2} f_H^*(x) e^{2\pi i x n} dx \\ &= \gamma_H^*(n) \sum_{j=-\infty}^{\infty} |G_j| \leq \infty. \end{aligned}$$

This justifies the use of Fubini's Theorem (Taylor (1973), Th.6.5) on the iterated integral (15) to reverse the order of integration and summation. Using the evenness of G and f_H^* , this yields

$$\begin{aligned} \gamma_H(n) &= \sum_{j=-\infty}^{\infty} G_j \int_{-1/2}^{1/2} f_H^*(x) \cos(2\pi x j) \cos(2\pi x n) dx \\ &= \sum_{j=-\infty}^{\infty} G_j \int_{-1/2}^{1/2} f_H^*(x) \frac{1}{2} \left(\cos(2\pi x(j-n)) + \cos(2\pi x(j+n)) \right) dx \\ &= \frac{1}{2} \sum_{j=-\infty}^{\infty} G_j (\gamma_H^*(j-n) + \gamma_H^*(j+n)) = \frac{1}{2} \left(\sum_{j=-\infty}^{\infty} G_j \gamma_H^*(n-j) + \sum_{j=-\infty}^{\infty} G_j \gamma_H^*(-j-n) \right) \\ &= \frac{1}{2} \left((G \star \gamma_H^*)(n) + (G \star \gamma_H^*)(-n) \right) = (G \star \gamma_H^*)(n), \end{aligned}$$

using the evenness of γ_H^* and $G \star \gamma_H^*$, and the existence of $G \star \gamma_H^*$ to justify the splitting of the sum. \square

Lemma 2. Assume $1 < \alpha < 2$ and let $a = \{|n|^\alpha : n \in \mathbf{Z}\}$. Let b be a symmetric sequence satisfying $\sum_{j=1}^{\infty} j^\alpha |b_j| < \infty$. Then $S_b = \sum_{j=-\infty}^{\infty} b_j$ and the symmetric sequence $c = a \star b$ exist, and $(c_n - S_b a_n) \xrightarrow{n \rightarrow \infty} 0$.

Proof. Since $\alpha > 1$, $\sum_{j=-\infty}^{\infty} |b_j| \leq |b_0| + 2 \sum_{j=1}^{\infty} j^\alpha |b_j| < \infty$, so b is absolutely summable and hence summable. Now consider c . Clearly $c_0 = \sum_{j=-\infty}^{\infty} |-j|^\alpha b_j$ exists by the assumptions on b , and for $n > 0$

$$\begin{aligned} |c_n| = |(a \star b)_n| &\leq \sum_{j=-\infty}^{-n} |n-j|^\alpha |b_j| + \sum_{j=-n+1}^{n-1} |n-j|^\alpha |b_j| + \sum_{j=n}^{\infty} |n-j|^\alpha |b_j| \\ &\leq \sum_{j=n}^{\infty} (2j)^\alpha |b_j| + \sum_{j=-n+1}^{n-1} |n-j|^\alpha |b_j| + \sum_{j=n}^{\infty} j^\alpha |b_j| < \infty. \end{aligned}$$

Since both a and b are symmetric, c_n also exists for $n < 0$, and so c exists and is symmetric.

For the last part, since $c_n - S_b a_n$ is symmetric in n we assume $n \geq 0$ and rewrite it as

$$\sum_{j=-\infty}^{\infty} |n-j|^\alpha b_j - n^\alpha \sum_{j=-\infty}^{\infty} b_j = n^\alpha b_0 + \sum_{j=1}^{\infty} (n+j)^\alpha b_j + \sum_{j=1}^{\infty} |n-j|^\alpha b_j - n^\alpha \sum_{j=-\infty}^{\infty} b_j = \sum_{j=1}^{\infty} T_n^j b_j$$

where $T_n^j := |n-j|^\alpha + (n+j)^\alpha - 2n^\alpha$, $n \geq 0$, $j > 0$. Noticing that $T_n^j = f_\alpha(n, j)$ from Lemma A1, we have that $T_n^j < T_0^j = 2j^\alpha$ for each fixed j , and so

$$|c_n - S_b a_n| \leq \sum_{j=1}^N |T_n^j| |b_j| + \sum_{j=N+1}^{\infty} |T_n^j| |b_j| < \sum_{j=1}^N |T_n^j| |b_j| + 2 \sum_{j=N+1}^{\infty} j^\alpha |b_j|.$$

Now given any $\varepsilon > 0$, a $N(\varepsilon) > 1$ can be found such that $\sum_{j=N+1}^{\infty} j^\alpha |b_j| < \varepsilon/4$. Next, since $T_n^j \xrightarrow{n \rightarrow \infty} 0$ for any fixed j (Lemma A1 below), there exists an $n_0(N)$ such that $\sum_{j=1}^N |T_n^j| |b_j| < \varepsilon/2$ when $n \geq n_0$. It follows that $|c_n - S_b a_n| < \varepsilon$ for $n \geq n_0$ and so $(c_n - S_b a_n) \xrightarrow{n \rightarrow \infty} 0$. \square

Lemma A1 Assume $1 < \alpha < 2$ and define $f_\alpha(x, y) := |x-y|^\alpha + (x+y)^\alpha - 2x^\alpha$ for $x \geq 0$, $y > 0$. For each y , $f_\alpha(\cdot, y)$ is positive, strictly decreasing, and $\lim_{x \rightarrow \infty} f_\alpha(x, y) = 0$.

Proof. Fix $y > 0$. We split the domain of $f_\alpha(\cdot, y)$ into two cases.

Let $x \geq y$. It follows that $f'_\alpha(\cdot, y) = \alpha f_{\alpha-1}(\cdot, y)$. Define $g(x) = x^\alpha$. Since $g'(x) = \alpha x^{\alpha-1}$ is strictly concave, $(x-y)^{\alpha-1} + (x+y)^{\alpha-1} < 2x^{\alpha-1}$ and so $f'_\alpha(\cdot, y) < 0$ and $f_\alpha(\cdot, y)$ is strictly decreasing. To prove $\lim_{x \rightarrow \infty} f_\alpha(x, y) = 0$, we apply the mean value theorem twice to g , and then once to g' , to obtain:

$$f_\alpha(x, y) = ((x+y)^\alpha - x^\alpha) - (x^\alpha - (x-y)^\alpha) \quad (16)$$

$$< \alpha y ((x+y)^{\alpha-1} - (x-y)^{\alpha-1}) \quad (17)$$

$$< 2\alpha(\alpha-1)y^2(x-y)^{\alpha-2} \quad (18)$$

(since g' is strictly increasing and g'' strictly decreasing), which tends to zero as $x \rightarrow \infty$.

Let $x < y$. In this case, the derivative with respect to x yields

$$f'_\alpha(x, y) = \alpha((x+y)^{\alpha-1} - (y-x)^{\alpha-1} - 2x^{\alpha-1}) \quad (19)$$

$$< \alpha((x+y)^{\alpha-1} - (y-x)^{\alpha-1} - (2x)^{\alpha-1}) \quad (20)$$

$$= \alpha(h_x(y) - h_x(x)) \quad (21)$$

where $h_x(y) = (x+y)^{\alpha-1} - (y-x)^{\alpha-1}$. Since the derivative of h_x is negative for $x > 0$, h_x is strictly decreasing. It follows that $f'_\alpha(\cdot, y) < 0$ and so $f_\alpha(\cdot, y)$ is likewise strictly decreasing.

Finally, since $f_\alpha(x, y)$ is decreasing for all $x \geq 0$ and tends to zero, it is positive. \square

Lemma 3. Let a, b be symmetric sequences and assume that $c := a \star b$ exists. Then $\mathbf{I}c$ exists, and if $(\mathbf{I}a) \star b$ exists, then $\mathbf{I}c = (\mathbf{I}a) \star b - ((\mathbf{I}a) \star b)_0$.

Proof. Since $(\mathbf{I}c)_n$ is a finite sum of elements of c , it exists for each n . Now

$$(\mathbf{I}c)_n = \sum_{k=0}^{n-1} \sum_{i=-k}^k \sum_{j=-\infty}^{\infty} a_j b_{i-j}$$

can be rewritten as $(\mathbf{I}c)_n = \sum_{j=-\infty}^{\infty} a_j H_n(j)$ where $H_n(j) := \sum_{k=0}^{n-1} \sum_{i=-k}^k b_{i-j}$, since a finite sum of convergent series is convergent. Since $(\mathbf{I}a)_{j-1} - 2(\mathbf{I}a)_j + (\mathbf{I}a)_{j+1} = a_{-j} + a_j = 2a_j$, we have

$$(\mathbf{I}c)_n = \sum_{j=-\infty}^{\infty} a_j H_n(j) = \frac{1}{2} \sum_{j=-\infty}^{\infty} ((\mathbf{I}a)_{j-1} - 2(\mathbf{I}a)_j + (\mathbf{I}a)_{j+1}) H_n(j) \quad (22)$$

$$= \frac{1}{2} \left(\sum_{j=-\infty}^{\infty} (\mathbf{I}a)_{j-1} H_n(j) - 2 \sum_{j=-\infty}^{\infty} (\mathbf{I}a)_j H_n(j) + \sum_{j=-\infty}^{\infty} (\mathbf{I}a)_{j+1} H_n(j) \right) \quad (23)$$

$$= \frac{1}{2} \sum_{j=-\infty}^{\infty} (\mathbf{I}a)_j (H_n(j+1) - 2H_n(j) + H_n(j-1)), \quad (24)$$

Step (23) is justified since each of the sums is convergent, because each can be written as a finite sum of series of the form $\sum_{j=-\infty}^{\infty} (\mathbf{I}a)_j b_{m-j}$ for some m , and this is just $((\mathbf{I}a) \star b)_m$ which exists by assumption. Now

$$H_n(j+1) - 2H_n(j) + H_n(j-1) = (H_n(j-1) - H_n(j)) - (H_n(j) - H_n(j+1)) \quad (25)$$

$$= \sum_{k=0}^{n-1} \left(\left(\sum_{i=-k-j+1}^{k-j+1} b_i - \sum_{i=-k-j}^{k-j} b_i \right) - \left(\sum_{i=-k-j}^{k-j} b_i - \sum_{i=-k-j-1}^{k-j-1} b_i \right) \right) \quad (26)$$

$$= \sum_{k=0}^{n-1} \left((b_{k-j+1} - b_{-k-j}) - (b_{k-j} - b_{-k-j-1}) \right) \quad (27)$$

$$= \sum_{k=0}^{n-1} (b_{k-j+1} - b_{k-j}) - \sum_{k=0}^{n-1} (b_{-k-j} - b_{-k-j-1}) \quad (28)$$

$$= (b_{n-j} - b_{-j}) - (b_{-j} - b_{-n-j}) = b_{n-j} + b_{-n-j} - 2b_{-j}. \quad (29)$$

The result then follows by substitution into (24), using the existence of $(\mathbf{I}a) \star b$ to justify splitting the sum, and finally by the symmetry of $(\mathbf{I}a)$ and b . \square

7.3 ACVF

Lemma 4. Assume $-1 \leq \alpha < 0$ and let a be the symmetric positive sequence $a_n = |n|^\alpha$, $n \neq 0$ and $a_0 > 0$. Let b be a symmetric sequence with $|b_0| < \infty$ for which there exists $\beta \in [0, 2]$ such that $\sum_{j=1}^{\infty} j^\beta |b_j| < \infty$ and $|b_n| = O(n^{-(\beta+1)})$. Then $S_b := \sum_{j=-\infty}^{\infty} b_j$ and the symmetric sequence $c := a \star b$ exist, and $c_n - S_b a_n = O(n^{\alpha-\beta})$ as $n \rightarrow \infty$.

Proof. We have $\sum_{j=-\infty}^{\infty} |b_j| \leq b_0 + 2 \sum_{j=1}^{\infty} j^\beta |b_j| < \infty$ so b is absolutely summable and therefore summable. Then S_b exists. Moreover, $|c_n| = |(a \star b)_n| \leq \sum_{j=-\infty}^{\infty} |b_j| |a_{n-j}| \leq |b_n| a_0 + \sum_{j=-\infty}^{\infty} |b_j| < \infty$. We conclude that c_n exists for each $n \in \mathbf{Z}$, and that c is symmetric by the symmetry of a and b . Define $T_n^j := a_{|n-j|} + a_{n+j} - 2a_n$, and using the symmetry of a and b rewrite $c_n - S_b a_n$ as:

$$(a \star b)_n - S_b a_n = a_n b_0 + \sum_{j=1}^{\infty} a_{|n-j|} b_j + \sum_{j=1}^{\infty} a_{n+j} b_j - S_b a_n = \sum_{j=1}^{\infty} T_n^j b_j. \quad (30)$$

To prove the last part of the theorem it suffices to consider $n \geq 0$, since c is symmetric, and as we are interested in large n asymptotics, we restrict to $n > 2$. The sum for $|c_n - S_b a_n|$ can be decomposed as

$$|c_n - S_b a_n| \leq \sum_{j=1}^{\lfloor n/2 \rfloor} |T_n^j| |b_j| + \sum_{j=\lfloor n/2 \rfloor + 1}^{2n} |T_n^j| |b_j| + \sum_{j=2n+1}^{\infty} |T_n^j| |b_j| =: A_n + B_n + C_n. \quad (31)$$

We shall show that each of A_n, B_n , and C_n are of order $O(n^{\alpha-\beta})$.

The definition of A_n implies $n > j > 0$, so Lemma A2 below applies to $T_n^j = f_\alpha(n, j)$, and implies the existence of a constant $K > 0$ such that $|T_n^j| \leq K j^2 (n-j)^{\alpha-2} < K j^2 (n/2)^{\alpha-2}$ when $j \leq n/2$. Thus

$$K^{-1} 2^{\alpha-2} A_n \leq n^{\alpha-2} \sum_{j=1}^{\lfloor n/2 \rfloor} j^{2-\beta} j^\beta |b_j| \leq n^{\alpha-2} n^{2-\beta} \sum_{j=1}^{\lfloor n/2 \rfloor} j^\beta |b_j| \leq n^{\alpha-\beta} \sum_{j=1}^{\infty} j^\beta |b_j| = O(n^{\alpha-\beta}). \quad (32)$$

For B_n , where $n \neq j$ and $n, j > 0$, we have $|T_n^j| < 2|n-j|^\alpha$, while $|T_n^n| = a_0 + (2^\alpha - 2)a_n = O(1)$. Then for sufficiently large n , by assumption there exists a $K > 0$ such that

$$\begin{aligned} B_n &\leq 2 \sum_{j=\lfloor n/2 \rfloor + 1}^{n-1} (n-j)^\alpha |b_j| + 2 \sum_{j=n+1}^{2n} (j-n)^\alpha |b_j| + |T_n^n| |b_n| \\ &< 2K \left(\frac{n}{2} \right)^{-(\beta+1)} \left(\sum_{j=\lfloor n/2 \rfloor + 1}^{n-1} (n-j)^\alpha + \sum_{j=n+1}^{2n} (j-n)^\alpha + |T_n^n|/2 \right) \\ &< 2^{\beta+3} K n^{-(\beta+1)} \sum_{j=1}^n j^\alpha + O(n^{-(\beta+1)}) \\ &< 2^{\beta+3} K n^{-(\beta+1)} \left(1 + \int_1^n x^\alpha dx \right) + O(n^{-(\beta+1)}) = O(n^{\alpha-\beta}). \end{aligned}$$

For C_n , where $j \geq 2n$, we have $T_n^j < 2n^\alpha$. Since $\sum_{j=2n+1}^{\infty} |b_j| \leq (2n)^{-\beta} \sum_{j=2n+1}^{\infty} j^\beta |b_j| = o(n^{-\beta})$, we get

$$C_n \leq \sum_{j=2n+1}^{\infty} 2n^\alpha |b_j| \leq 2n^\alpha \sum_{j=2n+1}^{\infty} |b_j| = o(n^{\alpha-\beta}). \quad (33)$$

Conclude that $|c_n - S_b a_n| = O(n^{\alpha-\beta})$ as $n \rightarrow \infty$. \square

Lemma A2 Assume $\alpha < 0$ and define $f_\alpha(x, y) := |x-y|^\alpha + (x+y)^\alpha - 2x^\alpha$ for $x > y > 0$. Then $f_\alpha(x, y)$ is positive and obeys $f_\alpha(x, y) < 2\alpha(\alpha-1)y^2(x-y)^{\alpha-2}$.

Proof. Since $x > y$ it follows that $f'_\alpha(\cdot, y) = \alpha f_{\alpha-1}(\cdot, y)$. Define $g(x) = x^\alpha$. Since $g'(x) = \alpha x^{\alpha-1}$ is strictly concave, $\alpha(x-y)^{\alpha-1} + \alpha(x+y)^{\alpha-1} < 2\alpha x^{\alpha-1}$ and so $f'_\alpha(\cdot, y) < 0$ and $f_\alpha(\cdot, y)$ is strictly decreasing. Now apply the mean value theorem twice to g , and then once to g' , to obtain:

$$\begin{aligned} f_\alpha(x, y) &= ((x+y)^\alpha - x^\alpha) - (x^\alpha - (x-y)^\alpha) \\ &< \alpha y((x-y)^{\alpha-1} - (x+y)^{\alpha-1}) \\ &< 2\alpha(\alpha-1)y^2(x-y)^{\alpha-2}. \end{aligned}$$

since g' is strictly increasing and g'' strictly decreasing. \square

Details of the proof of Corollary 2 We explain here why $\varphi(x) = f_H(x) - f_H^*(x) = O(x^{-2H+3})$ as $x \rightarrow 0$. Calculate the first few derivatives of the analytic function $x \mapsto (\sin(x)/x)^{-2H+1}$ (set to 1 at $x = 0$) and expand in a Taylor series around the origin to find that $(\sin(x)/x)^{-2H+1} = 1 + O(x^2)$. It follows that

$\sin(x)^{-2H+1} = x^{-2H+1} + O(x^{-2H+3})$ for $x \neq 0$. The function h is assumed three times (continuously) differentiable. Symmetry implies $h'(0) = 0$ so that by Taylors theorem, $h(x) = h(0) + O(x^2)$. Thus

$$h(x) \sin(\pi x)^{-2H+1} = h(0) \pi^{-2H+1} x^{-2H+1} + O(x^{-2H+3}), \quad x \neq 0,$$

while it can be shown that

$$(1 - \cos(2\pi x))x^{-2H-1} = 2\pi^2 x^{-2H+1} + O(x^{-2H+3}), \quad x \neq 0,$$

and

$$(1 - \cos(2\pi x)) \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} |\pi j + \pi x|^{-2H-1} = O(x^2).$$

Then

$$\begin{aligned} f_H(x) - f_H^*(x) &= \{h(x)(2 \sin(\pi x))^{-2H+1}\} \\ &\quad - \{h(0)2^{-2H}\pi^{-2H-1}(1 - \cos(2\pi x))x^{-2H-1}\} + O(x^2) \\ &= \{h(0)\pi^{-2H+1}2^{-2H+1}x^{-2H+1} + O(x^{-2H+3})\} \\ &\quad - \{h(0)2^{-2H+1}\pi^{-2H+1}x^{-2H+1} + O(x^{-2H+3})\} + O(x^2) \\ &= O(x^{-2H+3}). \end{aligned}$$

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